On the Vortex Theory of Screw Propellers.

By SYDNEY GOLDSTEIN, M.A., Ph.D., St. John's College, Cambridge, and the Kaiser Wilhelm Institut für Strömungsforschung, Göttingen.

(Communicated by L. Prandtl, For.Mem.R.S.—Received January 21, 1929.)

1. *Introduction.*

The vortex-theory of screw propellers develops along similar lines to aerofoil theory. There is circulation of flow round each blade; this circulation vanishes at the tip and the root. The blade may be replaced by a bound vortex system, which, for the sake of simplicity, may be taken, as a first approximation, to be a bound vortex line. The strength of the vortex at any point is equal to Γ, the circulation round the corresponding blade section. From every point of this bound vortex spring free, trailing vortices, whose strength per unit length is $-\partial\Gamma/\partial r$, where r is distance from the axis of the screw. When the interference flow of this vortex system is small compared with the velocity of the blades, the trailing vortices are approximately helices, and together build a helical or screw surface.* Part of the work supplied by the motor is lost in producing the trailing vortex system. When the distribution of Γ along the blade is such that, for a given thrust, the energy so lost per unit time is a minimum, then the flow far behind the screw is the same as if the screw surface formed by the trailing vortices was rigid, and moved backwards in the direction of its axis with a constant velocity, the flow being that of classical hydrodynamics in an inviscid fluid, continuous, irrotational, and without circulation.† The circulation round any blade section is then equal to the discontinuity in

* This system is unstable, and at a sufficient distance behind the propeller the vorticity is mainly concentrated into as many helical vortices as there are blades, with radii somewhat less than that of the original system, and into a straight line vortex along the axis. The strength of each helical vortex is nearly equal to the maximum value of the circulation round a blade section, and the strength of the straight line vortex is equal and opposite to the total strength of all the helical vortices.

† Betz, 'Göttinger Nachr.,' pp. 193–213 (1919); reprinted in 'Vier Abhandlungen zur Hydrodynamik und Aerodynamik,' L. Prandtl und A. Betz, Göttingen, 1927, where a selected bibliography is given. The theory takes no account of the finite size of the boss, or of the influence of compressibility of the fluid at large tip speeds. The approximation that the trailing vortices form regular helices is equivalent to neglecting the contraction of the slip stream, and is valid only for small values of the thrust coefficient. Also, in finding the most favourable distribution, the energy losses arising from the profile drag of the blade sections are not taken into account.

the velocity potential at the corresponding point of the screw surface. Further, for symmetrical screws, the interference flow at the blade is half that at the corresponding point of the screw surface far behind the propeller.*

An approximate solution for the irrotational motion of a screw surface in an inviscid fluid was given by Prandtl.† The accuracy of the approximation increases with the number of blades and with the ratio of the tip speed to the velocity of advance, but for given values of these numbers we have no means of estimating the error, since the exact solution of the problem has not yet been found. It is the main object of this work to find the exact solution.

We consider first the important case of the two-bladed propeller; formulæ for any number of blades are given later. Finally, the application of the results is briefly considered.

2. *Formulation of the Potential Problem.*

Let R be the radius, ω the angular velocity, and v the velocity of advance of a two-bladed propeller. Let ε be the angle between 0 and $\frac{1}{2}\pi$ whose tangent is $v/r\omega$, where r is the distance from the axis of rotation. The equation to the screw surface to be considered is

$$\theta - \omega z/v = 0 \text{ or } \pi, \quad 0 \leqslant r \leqslant \mathrm{R}, \tag{1}$$

where r, θ and z are cylindrical polar co-ordinates. The axis of z is along the axis of the screw surface, with its positive direction away from the propeller.‡ If w is the velocity of the screw surface in the direction of its axis, and ϕ the potential, such that grad ϕ is the fluid velocity, the boundary conditions are that

$$w \cos \varepsilon = \frac{\partial \phi}{\partial z} \cos \varepsilon - \frac{\partial \phi}{r\,\partial \theta} \sin \varepsilon, \tag{2}$$

or

$$\omega r w = \omega r \frac{\partial \phi}{\partial z} - v \frac{\partial \phi}{r\,\partial \theta}, \tag{3}$$

for $\theta - \omega z/v = 0$ or π and $0 \leqslant r \leqslant \mathrm{R}$, and that grad ϕ should vanish for $r = \infty$.

The geometrical conditions are such that the fluid velocity is a function of r and $\theta - \omega z/v$ only. This does not by itself restrict ϕ to be a function of r and $\theta - \omega z/v$ only, since constant multiples of θ only and of z only may

* Betz, *loc. cit.*

† 'Göttinger Nachr.,' pp. 213–217 (1919); reprinted in the 'Vier Abhandlungen.'

‡ Since we are considering the motion far behind the propeller, the screw surface may be taken as extending to infinity in both directions along the axis of z.

occur. But no term in θ or z may occur in the expression for ϕ for $r > \mathrm{R}$, since the second would give finite velocity for infinite r, and the first would give circulation round the axis. Conditions of continuity now suffice to show that no term in θ or z may occur in the expression for ϕ for $r < \mathrm{R}$. Hence ϕ is a function of r and ζ, where

$$\zeta = \theta - \omega z/v. \tag{4}$$

The boundary condition (3) then becomes

$$\frac{\partial \phi}{\partial \zeta} = -\frac{\omega^2 r^2}{v^2 + \omega^2 r^2}\frac{wv}{\omega} \tag{5}$$

for $\zeta = 0$ or π and $0 \leqslant r \leqslant \mathrm{R}$.

It is convenient to take wv/ω temporarily equal to 1. Also, let

$$\mu = \omega r/v. \tag{6}$$

Then (5) becomes

$$\frac{\partial \phi}{\partial \zeta} = -\frac{\mu^2}{1 + \mu^2} \tag{7}$$

for $\zeta = 0$ or π, and $0 \leqslant r \leqslant \mathrm{R}$.

The differential equation satisfied by ϕ, namely $\nabla^2 \phi = 0$, can then be written

$$\left(\mu \frac{\partial}{\partial \mu}\right)^2 \phi + (1 + \mu^2)\frac{\partial^2 \phi}{\partial \zeta^2} = 0. \tag{8}$$

In addition to satisfying (7), ϕ must be a single-valued function of position, and its derivatives must vanish when r is infinite. It must also be continuous everywhere except at the screw surface.

3. *Solution of the Potential Problem.*

3.1. *The Form of the Solution.**—It is not difficult to see that the conditions of the problem are such that ϕ is an odd function of ζ and of $\frac{1}{2}\pi - \zeta$, and since it is a single-valued function of position, continuous for $r > \mathrm{R}$, it can be expanded, for $r > \mathrm{R}$, in a series of sines of even multiples of ζ. Assuming such an expansion, differentiating term by term, and substituting in 2 (8), we find that the coefficient of $\sin 2n\zeta$ must be a linear function of $\mathrm{I}_{2n}(2n\mu)$ and $\mathrm{K}_{2n}(2n\mu)$, where I_n and K_n are the modified Bessel functions. But

* The reasons for adopting the form and method of solution to be described will be clear to anyone who reads Appendices 1 and 2.

$I_{2n}(2n\mu)$ cannot occur, since grad ϕ must vanish when r, or μ, is infinite. Hence we may assume

$$\phi = \sum_{n=1}^{\infty} c_n \frac{K_{2n}(2n\mu)}{K_{2n}(2n\mu_0)} \sin 2n\zeta, \text{ for } r > R, \tag{1}$$

where the c_n are constants to be determined,

$$\mu_0 = \omega R/v, \tag{2}$$

and $K_{2n}(2n\mu_0)$ has been inserted in the denominator for future convenience. We shall assume that the expansion holds also for $r = R$.

For $0 \leqslant r \leqslant R$, we put

$$\phi = -\frac{\mu^2}{1+\mu^2}\zeta + \phi_1, \tag{3}$$

so that

$$\left(\mu\frac{\partial}{\partial\mu}\right)^2 \phi_1 + (1+\mu^2)\frac{\partial^2\phi_1}{\partial\zeta^2} = \left(\mu\frac{d}{d\mu}\right)^2\left(\frac{\mu^2}{1+\mu^2}\right)\zeta, \tag{4}$$

and

$$\frac{\partial\phi_1}{\partial\zeta} = 0 \tag{5}$$

at $\zeta = 0$ and π.

Consider first the region for which $0 \leqslant \zeta \leqslant \pi$. For this region

$$\zeta = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)\zeta}{(2m+1)^2}. \tag{6}$$

If we assume

$$\phi_1 = f_0(\mu) + \sum_{m=0}^{\infty} f_m(\mu)\cos(2m+1)\zeta, \tag{7}$$

differentiate term by term, and substitute in (4), we find that

$$\left(\mu\frac{d}{d\mu}\right)^2 f_0(\mu) = \tfrac{1}{2}\pi\left(\mu\frac{d}{d\mu}\right)^2\left(\frac{\mu^2}{1+\mu^2}\right), \tag{8}$$

and

$$\left(\mu\frac{d}{d\mu}\right)^2 f_m(\mu) - (2m+1)^2(1+\mu^2) f_m(\mu) = -\frac{4}{\pi}\frac{1}{(2m+1)^2}\left(\mu\frac{d}{d\mu}\right)^2\left(\frac{\mu^2}{1+\mu^2}\right). \tag{9}$$

ϕ must be finite at $r = 0$, and so contains no term in $\log\mu$. Hence

$$f_0(\mu) = \tfrac{1}{2}\pi\,\mu^2/(1+\mu^2). \tag{10}$$

If we put

$$f_m(\mu) = -\frac{4}{\pi}\frac{1}{(2m+1)^2}\left\{\frac{\mu^2}{1+\mu^2} - g_m(\mu)\right\}, \tag{11}$$

then

$$\left(\mu\frac{d}{d\mu}\right)^2 g_m(\mu) - (2m+1)^2(1+\mu^2)\, g_m(\mu) = -(2m+1)^2\mu^2, \tag{12}$$

a particular solution of which is $S_{1, 2m+1}(\overline{2m+1}\, i\mu)$, where $S_{1,\nu}(z)$ is a Lomme function as defined in Watson's 'Bessel Functions,' § 10.71, p. 347. The general solution contains, in addition, terms in $I_{2m+1}(\overline{2m+1}\,\mu)$ and $K_{2m+1}(\overline{2m+1}\,\mu)$.

Corresponding to equation 10.71 (3) of Watson's 'Bessel Functions' we have

$$S_{1,\nu}(iz) = \frac{\nu\pi}{2\sin\frac{1}{2}\nu\pi} I_\nu(z) + \nu e^{\frac{1}{2}\nu\pi i} K_\nu(z) - t_{1,\nu}(z), \tag{13}$$

where

$$t_{1,\nu}(z) = \frac{z^2}{2-\nu^2} + \frac{z^4}{(2^2-\nu^2)(4^2-\nu^2)} + \frac{z^6}{(2^2-\nu^2)(4^2-\nu^2)(6^2-\nu^2)} + \dots$$
$$+ \frac{z^{2m}}{(2^2-\nu^2)(4^2-\nu^2)\dots(4m^2-\nu^2)} + \dots \tag{14}$$

In order to have a function which is real when z is real, and which (for positive ν) has no singularity at $z = 0$, we define the function $T_{1,\nu}(z)$ as being equal to the expression on the right of (13) without the term in $K_\nu(z)$. Thus

$$T_{1,\nu}(z) = \frac{\nu\pi}{2\sin\frac{1}{2}\nu\pi} I_\nu(z) - t_{1,\nu}(z) \tag{15}$$

where $t_{1,\nu}(z)$ is given by (14).* Then $T_{1, 2m+1}(\overline{2m+1}\,\mu)$ is also a particular solution of (12). Its value is found by putting $(2m+1)$ for ν and $(2m+1)\mu$ for z in (14) and (15). As the general solution of (12) which remains finite on the axis, when μ is zero, we find then

$$g_m(\mu) = T_{1, 2m+1}(\overline{2m+1}\,\mu) + b_m I_{2m+1}(\overline{2m+1}\,\mu), \tag{16}$$

where the b_m are constants to be determined.

If now in (3) we substitute for ζ from (6), for ϕ_1 from (7), for f_0 from (10) and for f_m from (11), we find

$$\phi = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{g_m(\mu)}{(2m+1)^2}\cos(2m+1)\zeta$$

$$= \sum_{m=0}^{\infty}\left[\frac{4}{\pi}\frac{T_{1, 2m+1}(\overline{2m+1}\,\mu)}{(2m+1)^2} + a_m \frac{I_{2m+1}(\overline{2m+1}\,\mu)}{I_{2m+1}(\overline{2m+1}\,\mu_0)}\right]\cos(2m+1)\zeta, \tag{17}$$

where the a_m are new constants to be determined, and μ_0 is $\omega R/v$, as before. This expression holds for $0 \leqslant r \leqslant R$, and for $0 \leqslant \zeta \leqslant \pi$. Also ϕ is an odd

* The expression given in (15) for $T_{1,\mu}(z)$ is indeterminate when ν is an even integer. This happens in the case of the four-bladed propeller, and will be discussed later.

function of ζ, and for $-\pi \leqslant \zeta \leqslant 0$, we have the same expression with the opposite sign. The discontinuity in ϕ at $\zeta = 0$ or π is thus

$$[\phi] = \frac{8}{\pi} \sum_{m=0}^{\infty} \frac{\mathrm{T}_{1,\,2m+1}\,(\overline{2m+1}\,\mu)}{(2m+1)^2} + 2 \sum_{m=0}^{\infty} a_m \frac{\mathrm{I}_{2m+1}\,(\overline{2m+1}\,\mu)}{\mathrm{I}_{2m+1}\,(\overline{2m+1}\,\mu_0)}. \qquad (18)$$

It remains to determine the constants a_m in (17) and c_m in (1) from the conditions of continuity at $r = \mathrm{R}$. But to do this we shall require to have the numerical values of the T functions and their derivatives. These are not tabulated, and so we must find formulæ for computing them. It will be convenient to discuss at the same time the numerical evaluation of the first term of (18), which is independent of μ_0. This term gives the result for infinite μ_0, that is, infinite R ; the second term gives the influence of the edge.

3.2. *Discussion of the T Functions.*—The formulæ defining T, given in 3.1 (14) and (15) (p. 444), are suitable for computation for small μ. Again, $\mathrm{T}_{1,\nu}\,(z)$ and $\mathrm{S}_{1,\nu}\,(iz)$ differ only by a multiple of $\mathrm{K}_\nu\,(z)$, which is exponentially small for large values of the real part of z. Hence for $|\arg z| < \frac{1}{2}\pi$, $\mathrm{T}_{1,\nu}\,(z)$ has the same asymptotic expansion as $\mathrm{S}_{1,\nu}\,(iz)$, namely

$$\mathrm{T}_{1,\nu}\,(z) \sim 1 - \nu^2/z^2 + \nu^2(\nu^2 - 2^2)/z^4 - \nu^2(\nu^2-2^2)(\nu^2-4^2)/z^6 + \ldots \qquad (1)$$

(Watson's 'Bessel Functions,' § 10.75, p. 351).

Another formula for $\mathrm{T}_{1,\,2m+1}\,(\overline{2m+1}\,\mu)$ can be found by substituting

$$\mathrm{T}_{1,2m+1}\,(\overline{2m+1}\,\mu) = \tau_0\,(\mu) + \tau_2\,(\mu)/(2m+1)^2 + \tau_4\,(\mu)/(2m+1)^4 + \ldots \qquad (2)$$

in the equation 3.1 (12) and equating to zero the coefficients of powers of $(2m+1)$. This gives

$$\tau_0 = \mu^2/(1+\mu^2), \qquad (3)$$

and

$$\tau_{r+2} = \frac{1}{1+\mu^2}\left(\mu\frac{d}{d\mu}\right)^2 \tau_r, \qquad (4)$$

so that

$$\tau_2 = 4\mu^2(1-\mu^2)/(1+\mu^2)^4, \qquad (5)$$

$$\tau_4 = 16\mu^2\{1 - 14\mu^2 + 21\mu^4 - 4\mu^6\}/(1+\mu^2)^7, \qquad (6)$$

and

$$\tau_6 = 64\mu^2\{1 - 75\mu^2 + 603\mu^4 - 1065\mu^6 + 460\mu^8 - 36\mu^{10}\}/(1+\mu^2)^{10}. \qquad (7)$$

(The expansion is probably convergent for some values of μ_0 and asymptotic for others.) It is not immediately obvious that the formal solution of 3.1 (12) so obtained represents $T_{1,\,2m+1}\,(\overline{2m+1}\,\mu)$; that this is so becomes clear on comparing numerical values calculated from 3.1 (14) and (15), or from (1) above, with those calculated from (2).

By using the formulæ 3.1 (14), (15), 3.2 (1), (2) to (7) we can now tabulate $T_{1,\,2m+1}\,(\overline{2m+1}\,\mu)$. We find that it is always very near to $\mu^2/(1+\mu^2)$, and for any given value of μ approaches nearer and nearer this value as m increases. If we put

$$\frac{\mu^2}{1+\mu^2} - T_{1,\,2m+1}\,(\overline{2m+1}\,\mu) = F_{2m+1}\,(\mu), \tag{8}$$

the values of $F_{2m+1}\,(\mu)$ for various values of m and μ are given in the table below.

Table I.

μ.	$F_1\,(\mu)$.	$F_3\,(\mu)$.	$F_5\,(\mu)$.	$G\,(\mu)$.
0·2	−0·1060	−0·0156	−0·0055	0·1260
0·4	−0·1287	−0·0271	−0·0110	0·2450
0·6	−0·1052	−0·0231	−0·0096	0·3524
0·8	−0·0656	−0·0125	−0·0049	0·4447
1·0	−0·0316	−0·0026	−0·0005	0·5259
1·2	−0·0039	+0·0040	+0·0021	0·5930
1·4	+0·0140	+0·0075	0·0034	0·6501
1·6	+0·0251	0·0087	0·0036	0·6979
1·8	0·0311	0·0085	0·0034	0·7381
2·0	0·0336	0·0080	0·0030	0·7720
2·5	0·0316	0·0048	0·0020	0·8360
3·0	0·0254	0·0035	0·0011	0·8791
3·5	0·0192	0·0021	0·0007	0·9087
4·0	0·0141	0·0014	0·0005	0·9296
4·5	0·0102	0·0009	0·0003	0·9446
5·0	0·0073	0·0006	0·0002	0·9555
6·0	0·0039	0·0003	0·0001	0·9698
7·0	0·0021	0·0002	0·0001	0·9783
8·0	0·0012	0·0001	—	0·9836
9·0	0·0007	0·0001	—	0·9872

The last figure in the entries in this table may be incorrect.

Then

$$\frac{8}{\pi}\sum_{m=0}^{\infty}\frac{T_{1,\,2m+1}\,(\overline{2m+1}\,\mu)}{(2m+1)^2} = \frac{8}{\pi}\,\frac{\mu^2}{1+\mu^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} - \frac{8}{\pi}\sum_{m=0}^{\infty}\frac{F_{2m+1}\,(\mu)}{(2m+1)^2}$$

$$= \pi\left\{\frac{\mu^2}{1+\mu^2} - \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{F_{2m+1}\,(\mu)}{(2m+1)^2}\right\}. \tag{9}$$

The expression in curly brackets is also tabulated in Table I, where it is denoted by $G(\mu)$. Since the F are small, and decrease with m, the tabulation is rapid.

The values of $G(\mu)$ for $\mu = 2{\cdot}8$, $3{\cdot}8$, $4{\cdot}8$, $5{\cdot}5$, $5{\cdot}8$, $6{\cdot}5$, $6{\cdot}8$, $7{\cdot}5$, $7{\cdot}8$, $8{\cdot}5$, $8{\cdot}8$, $9{\cdot}5$ and $9{\cdot}8$ will be required later. They were found by interpolating in tables of $\mu^2/(1 + \mu^2) - G(\mu)$, etc.

This completes the discussion of the first term of 3.1 (18) (p. 445). To evaluate the second term we shall require the values of the derivatives of the T functions, formulæ for which are obtained by differentiating the formulæ for the T functions themselves. The approach of $(2m + 1)\, T'_{1,2m+1}\,(\overline{2m+1}\,\mu)$ is to $2\mu/(1 + \mu^2)^2$.

A few values are given below for exhibition purposes. (The dash denotes differentiation, so that $T'_{1,\nu}(z)$ is $dT_{1,\nu}(z)/dz$).

$$T'_{1,1}(5) = 0{\cdot}0185, \quad 3T'_{1,3}(15) = 0{\cdot}0152, \quad 2\,.\,5/(1 + 25)^2 = 0{\cdot}0148.$$

$$T'_{1,1}(3) = 0{\cdot}0723, \quad 3T'_{1,3}(9) = 0{\cdot}0630, \quad 2\,.\,3/(1 + 9)^2 = 0{\cdot}0600.$$

$$T'_{1,1}(2) = 0{\cdot}1543, \quad 3T'_{1,3}(6) = 0{\cdot}1649, \quad 2\,.\,2/(1 + 4)^2 = 0{\cdot}1600.$$

3.3. *The Determination of the Constants a_m.*—We must now turn to the determination of the constants a_m in 3.1 (17) and (18) (pp. 444–5) from the conditions of continuity at $r = R$. The conditions are the continuity of ϕ and of $\partial\phi/\partial r$. At $r = R$, and $\zeta = 0$ or π, singularities are to be expected, and so we cannot obtain analytical conditions for continuity there. Also ϕ is an odd function of ζ. Hence it is sufficient to ensure continuity for $0 < \zeta < \pi$, and continuity for $-\pi < \zeta < 0$ will automatically follow. But in the range $0 < \zeta < \pi$,

$$\cos(2m + 1)\,\zeta = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2 - (2m + 1)^2} \sin 2n\zeta. \tag{1}$$

We substitute this into 3.1 (17), re-arrange the resulting double series, and equate the coefficient of $\sin 2n\zeta$ so obtained to the coefficient of $\sin 2n\zeta$ in 3.1 (1) (p. 443), for $\mu = \mu_0$. We then differentiate both 3.1 (1) and 3.1 (17) term by term with respect to μ and repeat the process. This gives us the equations

$$\sum_{m=0}^{\infty} \left\{\frac{4}{\pi} \frac{T_{1,2m+1}}{(2m + 1)^2} + a_m\right\} \frac{8n}{4n^2 - (2m + 1)^2} = \pi c_n, \tag{2}$$

and

$$\sum_{m=0}^{\infty} \left\{\frac{4\ \ T'_{1,2m+1}}{\pi\,(2m + 1)^2} + a_m \frac{I'_{2m+1}}{I_{2m+1}}\right\} \frac{4\,(2m + 1)}{4n^2 - (2m + 1)^2} = \pi c_n \frac{K'_{2n}}{K_{2n}}. \tag{3}$$

The argument of the functions $T_{1,2m+1}$, $T'_{1,2m+1}$, I_{2m+1}, and I'_{2m+1} is $(2m + 1)\,\mu_0$; that of K_{2n} and K'_{2n} is $2n\mu_0$. Dashes denote differentiations, as before. Both equations hold for all integral values of n. Eliminating c_n gives us the equation

$$\frac{\pi}{4}\sum_{m=0}^{\infty}\frac{a_m\left\{(2m+1)\frac{\mathrm{I}'_{2m+1}}{\mathrm{I}_{2m+1}}-2n\frac{\mathrm{K}'_{2n}}{\mathrm{K}_{2n}}\right\}}{4n^2-(2m+1)^2}=$$

$$2n\frac{\mathrm{K}'_{2n}}{\mathrm{K}_{2n}}\sum_{m=0}^{\infty}\frac{\mathrm{T}_{1,2m+1}}{(2m+1)^2[4n^2-(2m+1)^2]}-\sum_{m=0}^{\infty}\frac{(2m+1)\mathrm{T}'_{1,2m+1}}{(2m+1)^2[4n^2-(2m+1)^2]}. \quad (4)$$

We have then an infinite number of linear equations with an infinite number of unknowns. The standard method of solving is to find successive approximations—to solve the first equation with $a_1 = a_2 = a_3 = \ldots = 0$, then the first two equations with $a_2 = a_3 = \ldots = 0$, and so on. But we may anticipate from the singularity in ϕ at the edges ($r = \mathrm{R}$, $\zeta = 0$ or π) that the convergence will be very slow. We have to solve for a whole set of values of μ_0, and the work would be prohibitive and the results inaccurate. Fortunately we can escape this difficulty.

We have seen that, so long as μ_0 is not too small, $\mathrm{T}_{1,2m+1}(\overline{2m+1}\,\mu_0)$ is nearly equal to $\mu_0^2/(1+\mu_0^2)$, and that $(2m+1)\mathrm{T}'_{1,2m+1}$ is small compared with $\mathrm{T}_{1,2m+1}$; also $(2m+1)\mathrm{I}'_{2m+1}/\mathrm{I}_{2m+1} - 2n\mathrm{K}'_{2n}/\mathrm{K}_{2n}$ is nearly equal to $(1+\mu_0^2)^{\frac{1}{2}}(2m+1+2n)$, and $\mathrm{K}'_{2n}/\mathrm{K}_{2n}$ is nearly equal to $-(1+\mu_0^2)^{\frac{1}{2}}$.* Thus approximately our equations may be written

$$\frac{\pi}{4}\sum_{m=0}^{\infty}\frac{a_m}{2n-2m-1}=-2n\frac{\mu_0^2}{1+\mu_0^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2[4n^2-(2m+1)^2]}$$

$$=-\frac{\mu_0^2}{1+\mu_0^2}\frac{\pi^2}{16n}, \quad (5)$$

or

$$\sum_{m=0}^{\infty}\frac{a_m}{2n-2m-1}=-\frac{\mu_0^2}{1+\mu_0^2}\frac{\pi}{4n}. \quad (6)$$

These equations may be solved exactly (Appendix 3), the solution being $a_m = -\dfrac{\mu_0^2}{1+\mu_0^2}\mathrm{A}_m$, where

$$\mathrm{A}_0 = 1,\quad 3\mathrm{A}_1 = \tfrac{1}{2},\quad 5\mathrm{A}_2 = \frac{1\cdot 3}{2\cdot 4},\quad 7\mathrm{A}_3 = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6},\ \ldots \quad (7)$$

We then put†

$$a_m = -\frac{\mu_0^2}{1+\mu_0^2}\mathrm{A}_m + \varepsilon_m, \quad (8)$$

* See the asymptotic approximations to $\mathrm{I}_n(nx)$ and $\mathrm{K}_n(nx)$ given by Nicholson, 'Phil. Mag.,' vol. 20, p. 938 (1910). But the approximations are much better than might be expected, especially for $(2m+1)\mathrm{I}'_{2m+1}/\mathrm{I}_{2m+1} - 2n\mathrm{K}'_{2n}/\mathrm{K}_{2n}$.

† The symbol ϵ is differently used in section 2.

substitute in (4), and solve numerically for ε_0, ε_1, ... in the manner described above. The ε are small; even for the lowest value of μ_0 to be considered, namely 2, the degree of accuracy we shall finally aim at is attained if ε_0 is found correct to two significant figures, with a possible error of 1 or 2 in the second figure, if ε_1 is found to one significant figure, and if the other ε are ignored. The convergence is slow, but not so slow that we cannot manage this with a reasonable amount of labour. The equations were solved numerically for $\mu_0 = 2$, 3 and 5, with the following results :—

$$\mu_0 = 2; \quad \varepsilon_0 = -0{\cdot}061, \ \varepsilon_1 = 0{\cdot}013. \tag{9}$$

$$\mu_0 = 3; \quad \varepsilon_0 = -0{\cdot}047, \ \varepsilon_1 = 0{\cdot}007. \tag{10}$$

$$\mu_0 = 5; \quad \varepsilon_0 = -0{\cdot}033, \ \varepsilon_1 = 0{\cdot}004. \tag{11}$$

It will be seen that, to one significant figure, $-\varepsilon_0$ is $0{\cdot}06$ when μ_0 is 2; it is $0{\cdot}05$ when μ_0 is 3, and $0{\cdot}03$ when μ_0 is 5. It was therefore taken, without any further ado, to be $0{\cdot}04$ when μ_0 is 4; to be $0{\cdot}02$ when μ_0 is 6, and $0{\cdot}01$ when μ_0 is 7; and the other ε were neglected for these values of μ_0.* For higher values of μ_0 all the ε may be neglected.

Thus we find the a_m to a sufficient approximation. The I functions can be found in tables, and the second term in 3.1 (18) easily calculated. Unless μ is very near indeed to μ_0, the terms decrease rapidly, and we require only the first few terms.

3.4. *The Distribution of Circulation.*—From 3.1 (18), 3.2 (9) and 3.3 (8)

$$[\phi]/\pi = \mathrm{G}(\mu) - \frac{2}{\pi}\sum_{m=0}^{\infty}\left(\frac{\mu_0^2}{1+\mu_0^2}\mathrm{A}_m - \varepsilon_m\right)\frac{\mathrm{I}_{2m+1}(\overline{2m+1}\,\mu)}{\mathrm{I}_{2m+1}(\overline{2m+1}\,\mu_0)}, \tag{1}$$

where G (μ) is the expression in curly brackets in 3.2 (9) and is tabulated in Table I, A_m is given in 3.3 (7) and ε_m in 3.3 (9), (10), (11) and the paragraph following.

As mentioned in the introduction, $[\phi]$ is equal to the circulation round the corresponding blade section. If now we restore the factor wv/ω, temporarily taken as 1 from the end of section 2 onwards, we have

$$\frac{\Gamma\omega}{\pi wv} = \mathrm{G}(\mu) - \frac{2}{\pi}\sum_{m=0}^{\infty}\left(\frac{\mu_0^2}{1+\mu_0^2}\mathrm{A}_m - \varepsilon_m\right)\frac{\mathrm{I}_{2m+1}(\overline{2m+1}\,\mu)}{\mathrm{I}_{2m+1}(\overline{2m+1}\,\mu_0)}. \tag{2}$$

* This will increase the possible error in the calculated value of $\Gamma\omega/\pi wv$ at $\mu = 3{\cdot}8$ when μ_0 is 4 to about 7 or 8 in the last figure. Consequently, the corresponding point in the graph may be displaced this amount if the curve can thereby be smoothed. The actual displacement was from $0{\cdot}334$ to $0{\cdot}330$.

There is no further difficulty in computing the expression on the right. The values for $\mu_0 = 2, 3, 4, \dots 9, 10$ are given in Table II below. These are the

Table II.—The Values of $\Gamma\omega/\pi wv$.

μ \ μ_0	2.	3.	4.	5.	6.	7.	8.	9.	10.
0·0	0	0	0	0	0	0	0	0	0
0·2	0·092	0·111	0·120	0·124	0·125	0·126	0·126	0·126	0·126
0·4	0·175	0·213	0·232	0·240	0·243	0·245	0·245	0·245	0·245
0·6	0·243	0·303	0·331	0·344	0·349	0·351	0·352	0·352	0·352
0·8	0·295	0·379	0·418	0·434	0·441	0·443	0·445	0·445	0·445
1·0	0·329	0·440	0·489	0·511	0·520	0·523	0·525	0·526	0·526
1·2	0·341	0·485	0·548	0·575	0·585	0·590	0·592	0·593	0·593
1·4	0·331	0·514	0·592	0·626	0·640	0·646	0·649	0·650	0·650
1·6	0·295	0·533	0·628	0·669	0·687	0·694	0·696	0·698	0·698
1·8	0·220	0·537	0·654	0·704	0·724	0·732	0·736	0·737	0·738
2·0	0	0·525	0·670	0·731	0·755	0·766	0·769	0·771	0·772
2·5		0·427	0·676	0·770	0·810	0·826	0·832	0·835	0·836
2·8		0·303	*	*	*	*	*	*	*
3·0		0	0·621	0·775	0·838	0·863	0·873	0·877	0·878
3·5			0·486	0·747	0·846	0·884	0·899	0·905	0·908
3·8			0·334	*	*	*	*	*	*
4·0			0	0·671	0·830	0·890	0·915	0·924	0·927
4·5				0·519	0·786	0·882	0·920	0·935	0·940
4·8				0·351	*	*	*	*	*
5·0				0	0·701	0·858	0·918	0·941	0·950
5·5					0·543	*	*	*	*
5·8					0·368	*	*	*	*
6·0					0	0·717	0·874	0·932	0·955
6·5						0·554	*	*	*
6·8						0·376	*	*	*
7·0						0	0·728	0·883	0·941
7·5							0·560	*	*
7·8							0·382	*	*
8·0							0	0·734	0·890
8·5								0·566	*
8·8								0·386	*
9·0								0	0·738
9·5									0·569
9·8									0·388
10·0									0

* Not calculated.

calculated points from which the full line curves in fig. 1 were drawn. It may be mentioned that in terms of the usual parameter J, equal to $v/n\mathrm{D}$, where n is the number of revolutions per unit time and D is the diameter, we have

$$\mu_0 = \pi/\mathrm{J}. \tag{3}$$

In view of recent and possible future developments, values of μ_0 as low as 2 were considered.

In general, the error in any entry in Table II should not exceed 2 or 3 in the last figure.

In the notation here used, Prandtl's approximate solution is

$$\frac{\Gamma\omega}{\pi w v} = \frac{2}{\pi}\frac{\mu^2}{1+\mu^2}\cos^{-1} e^{-f}, \tag{4}$$

where

$$f = (1 - \mu/\mu_0)(1 + \mu_0^2)^{\frac{1}{2}}. \tag{5}$$

This is shown by the broken lines in fig. 1.

3.5. *The Determination of the Flow.*—The potential for the flow for $r \leqslant R$, $0 \leqslant \zeta \leqslant \pi$, is given by 3.1 (17), where the a_m have been determined to a sufficient approximation. Restoring the factor wv/ω, we may write, using 3.2 (8)

$$\frac{\omega\phi}{wv} = \frac{4}{\pi}\frac{\mu^2}{1+\mu^2}\sum_{m=0}^{\infty}\frac{\cos(2m+1)\zeta}{(2m+1)^2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{F_{2m+1}(\mu)}{(2m+1)^2}\cos(2m+1)\zeta$$
$$- \sum_{m=0}^{\infty}\left(\frac{\mu_0^2}{1+\mu_0^2}A_m - \varepsilon_m\right)\frac{I_{2m+1}(\overline{2m+1}\,\mu)}{I_{2m+1}(\overline{2m+1}\,\mu_0)}\cos(2m+1)\zeta$$
$$= \frac{\mu^2}{1+\mu^2}(\tfrac{1}{2}\pi - \zeta) - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{F_{2m+1}(\mu)}{(2m+1)^2}\cos(2m+1)\zeta$$
$$- \sum_{m=0}^{\infty}\left(\frac{\mu_0^2}{1+\mu_0^2}A_m - \varepsilon_m\right)\frac{I_{2m+1}(\overline{2m+1}\,\mu)}{I_{2m+1}(\overline{2m+1}\,\mu_0)}\cos(2m+1)\zeta. \tag{1}$$

Hence

$$\frac{\omega}{wv}\frac{\partial\phi}{\partial\zeta} = -\frac{\mu^2}{1+\mu^2} + \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{F_{2m+1}(\mu)}{2m+1}\sin(2m+1)\zeta$$
$$+ \sum_{m=0}^{\infty}\left(\frac{\mu_0^2}{1+\mu_0^2}A_m - \varepsilon_m\right)\frac{I_{2m+1}(\overline{2m+1}\,\mu)}{I_{2m+1}(\overline{2m+1}\,\mu_0)}(2m+1)\sin(2m+1)\zeta. \tag{2}$$

Since $F_{2m+1}(\mu)$ is given in Table I, this is easily calculated. We notice that the first series in the last term of 3.1 (17) cannot be differentiated term by term with respect to ζ, since the resulting series does not converge.

Also

$$\frac{\omega}{wv}\frac{\partial\phi}{\partial\mu} = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{T'_{1,\,2m+1}(\overline{2m+1}\,\mu)}{2m+1}\cos(2m+1)\zeta$$
$$- \sum_{m=0}^{\infty}\left(\frac{\mu_0^2}{1+\mu_0^2}A_m - \varepsilon_m\right)\frac{I'_{2m+1}(\overline{2m+1}\,\mu)}{I_{2m+1}(\overline{2m+1}\,\mu_0)}(2m+1)\cos(2m+1)\zeta. \tag{3}$$

The first term approaches $\dfrac{2\mu}{(1+\mu^2)^2}(\tfrac{1}{2}\pi - \zeta)$. Since

$$I'_{2m+1}(\overline{2m+1}\,\mu_0)/I_{2m+1}(\overline{2m+1}\,\mu_0)$$

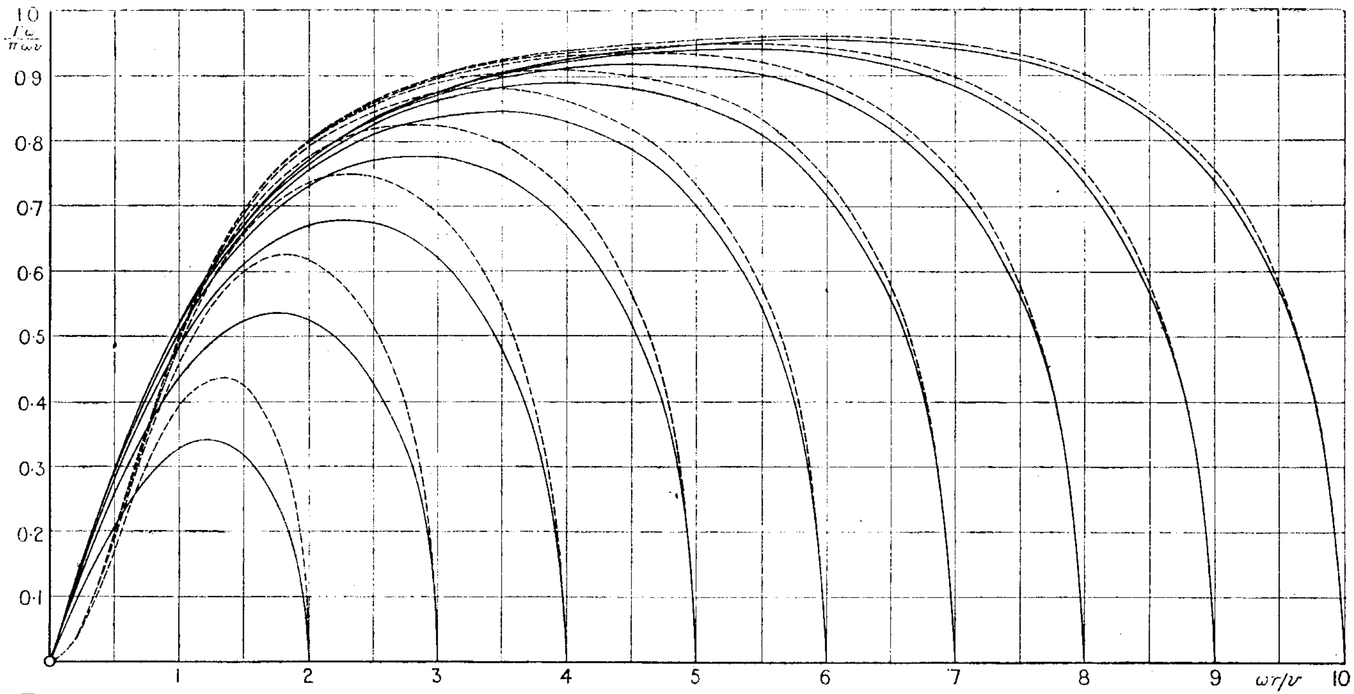

FIG. 1.—THE DISTRIBUTION OF CIRCULATION ALONG A PROPELLER BLADE, FOR A TWO-BLADED PROPELLER, WHEN THE ENERGY LOST IN THE SLIP STREAM IS A MINIMUM FOR A GIVEN THRUST.

The ordinates are $\Gamma\omega/\pi w v$, the abscissæ $\omega r/v$. Here Γ is the circulation at a distance r from the axis, ω the angular velocity and v the velocity of advance of the propeller. The flow far behind the propeller is the same as if the screw surface formed by the trailing vortices was rigid and moved backwards along its axis with velocity w. Also π is $3 \cdot 14 \ldots$. The curves are drawn for a series of values of $\omega R/v$, where R is the radius of the propeller. The value of $\omega R/v$ corresponding to any curve is the abscissa of the point, other than the origin, where the ordinate vanishes. The full line curves give the exact solution, the dotted curves Prandtl's approximate solution, drawn for the same value of w (that is, when the thrust coefficient is small, for the same efficiency).

tends, with increasing m, to become independent of m, we can easily see from 3.3 (7) (p. 448) that the second term becomes infinite at $\mu = \mu_0$, like a multiple of $(1 - \mu^2/\mu_0^2)^{-\frac{1}{2}}$.

If u_r, u_θ and u_z are the fluid velocities in the directions of r, θ and z increasing respectively, then

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\zeta} = \frac{\omega}{v}\frac{1}{\mu}\frac{\partial\phi}{\partial\zeta}, \tag{4}$$

$$u_z = -\frac{\omega}{v}\frac{\partial\phi}{\partial\zeta} \tag{5}$$

and

$$u_r = \frac{\omega}{v}\frac{\partial\phi}{\partial\mu}. \tag{6}$$

Hence the fluid velocities at any point are easily found from (2) and (3).

At the surface, u_θ and u_z are determined by the boundary condition alone, and are

$$u_\theta = -w\,\mu/(1+\mu^2), \quad u_z = w\,\mu^2/(1+\mu^2). \tag{7}$$

At the propeller the velocities are half as much, if the contraction of the slip-stream be neglected.

The potential for the flow for $r > \mathrm{R}$ is given by 3.1 (1) (p. 443), from which we see that the flow dies away rapidly as r increases. It is of some interest to find approximations for the constants c_n. This is done in Appendix 3, and the result is

$$c_m = \frac{\mu_0^2}{1+\mu_0^2}\frac{\mathrm{C}_m}{2m}, \tag{8}$$

where

$$\mathrm{C}_1 = \tfrac{1}{2}, \quad \mathrm{C}_2 = \frac{1\,.\,3}{2\,.\,4}, \quad \mathrm{C}_3 = \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}, \ldots \tag{9}$$

Expressions for the fluid velocity can be written down at once. Thus

$$u_z = -w\frac{\mu_0^2}{1+\mu_0^2}\left[\tfrac{1}{2}\frac{\mathrm{K}_2(2\mu)}{\mathrm{K}_2(2\mu_0)}\cos 2\zeta + \frac{1\,.\,3}{2\,.\,4}\frac{\mathrm{K}_4(4\mu)}{\mathrm{K}_4(4\mu_0)}\cos 4\zeta + \ldots\right], \tag{10}$$

and so on.

4. *The Solution for any Number of Blades.*

4.1. Let us suppose there are p blades. Then

$$\left(\mu\frac{\partial}{\partial\mu}\right)^2\phi + (1+\mu^2)\frac{\partial^2\phi}{\partial\zeta^2} = 0, \tag{1}$$

$$\frac{\partial\phi}{\partial\zeta} = -\frac{\mu^2}{1+\mu^2}\frac{wv}{\omega}\text{ for } r \leqslant \mathrm{R}, \text{ at } \zeta = 0, \frac{2\pi}{p}, \frac{4\pi}{p}, \ldots \frac{2(p-1)\pi}{p}, \tag{2}$$

and grad ϕ vanishes for $r = \infty$.

The solution for $r \leqslant R$, $0 \leqslant \zeta \leqslant 2\pi/p$ is

$$\frac{\omega\phi}{wv} = \frac{8}{p\pi} \sum_{m=0}^{\infty} \frac{T_{1,p(m+\frac{1}{2})}(\overline{m+\frac{1}{2}p\mu})}{(2m+1)^2} \cos(2m+1)\frac{p\zeta}{2}$$

$$- \frac{2}{p} \frac{\mu_0^2}{1+\mu_0^2} \sum_{m=0}^{\infty} A_m \frac{I_{p(m+\frac{1}{2})}(\overline{m+\frac{1}{2}p\mu})}{I_{p(m+\frac{1}{2})}(\overline{m+\frac{1}{2}p\mu_0})} \cos(2m+1)\frac{p\zeta}{2}, \quad (3)$$

where, approximately, the A_m have the same values as before, namely,

$$A_0 = 1, \quad 3A = \tfrac{1}{2}, \quad 5A_2 = \frac{1 \cdot 3}{2 \cdot 4}, \ldots \quad (4)$$

The exact values can be calculated in the same way as for two blades. The approximation becomes better and better the greater μ_0 and the greater the number of blades.

For $r > R$,

$$\frac{\omega\phi}{wv} = \frac{2}{p} \frac{\mu_0^2}{1+\mu_0^2} \sum_{n=1}^{\infty} \frac{C_n}{2n} \frac{K_{pn}(pn\mu)}{K_{pn}(pn\mu_0)} \sin pn\zeta, \quad (5)$$

where, approximately,

$$C_1 = \tfrac{1}{2}, \quad C_2 = \frac{1 \cdot 3}{2 \cdot 4}, \quad C_3 = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}, \ldots \quad (6)$$

As before, $T_{1,p(m+\frac{1}{2})}(\overline{m+\frac{1}{2}p\mu})$ is near to $\mu^2/(1+\mu^2)$. We put

$$\frac{\mu^2}{1+\mu^2} - T_{1,p(m+\frac{1}{2})}(\overline{m+\tfrac{1}{2}p\mu}) = F_{p,2m+1}(\mu), \quad (7)$$

so that

$$\sum_{m=0}^{\infty} \frac{T_{1,p(m+\frac{1}{2})}(\overline{m+\frac{1}{2}p\mu})}{(2m+1)^2} = \frac{\pi^2}{8} \frac{\mu^2}{1+\mu^2} - \sum_{m=0}^{\infty} \frac{F_{p,2m+1}(\mu)}{(2m+1)^2}, \quad (8)$$

and we find for the distribution of circulation

$$\frac{p\Gamma\omega}{2\pi wv} = \frac{\mu^2}{1+\mu^2} - \frac{8}{\pi^2} \sum_{m=0}^{\infty} \frac{F_{p,2m+1}(\mu)}{(2m+1)^2}$$

$$- \frac{2}{\pi} \frac{\mu_0^2}{1+\mu_0^2} \sum_{m=0}^{\infty} A_m \frac{I_{p(m+\frac{1}{2})}(\overline{m+\frac{1}{2}p\mu})}{I_{p(m+\frac{1}{2})}(\overline{m+\frac{1}{2}p\mu_0})}. \quad (9)$$

The A_m are given in (4); the I functions are the modified Bessel functions, defined as in Watson's 'Bessel Functions.' The F are related to the T functions by (7); to calculate the T functions we have formulæ 3.1 (14) and (15) (p. 444), and formula 3.2 (1) (p. 445). Also

$$T_{1,p(m+\frac{1}{2})}\,(\overline{m+\tfrac{1}{2}p\mu}) = \tau_0(\mu) + \frac{4\tau_2(\mu)}{p^2(2m+1)^2} + \frac{16\tau_4(\mu)}{p^4(2m+1)^4} + \dots, \qquad (10)$$

where

$$\tau_0 = \mu^2/(1+\mu^2), \qquad (11)$$

and

$$\tau_{r+2} = \frac{1}{1+\mu^2}\left(\mu\frac{d}{d\mu}\right)^2 \tau_r, \qquad (12)$$

so that the τ are the same as in 3.2 (3) to (7) (p. 445).

The only new point that arises for a three-bladed propeller is that the Bessel functions $I_{3/2}$, $I_{9/2}$, ... are not tabulated.* There are simple finite expressions for these functions, and to calculate the value for any one value of the argument is not difficult. But to make even a rough table involves a not inconsiderable amount of work. The numerical work for the three-bladed propeller is therefore left over for the present.

4.2. *The Four-bladed Propeller.*—For a four-bladed propeller we have to calculate $T_{1,\nu}(z)$, where ν is an even integer. The analytical peculiarity mentioned in the footnote on p. 444 enters, the expression 3.1 (15) for $T_{1,\nu}(z)$ becoming indeterminate. However, the asymptotic expansion 3.2 (1) (p. 445) terminates, and is equal to† $S_{1,\nu}(iz)$ or $T_{1,\nu}(z) + (-1)^{\frac{1}{2}\nu}\,\nu K_\nu(z)$, so that $T_{1,\nu}(z)$ can be calculated from the formula

$$T_{1,\nu}(z) = 1 - \nu^2/z^2 + \nu^2(\nu^2-2^2)/z^4 - \nu^2(\nu^2-2^2)(\nu^2-4^2)\,z^6 + \dots$$
$$- (-1)^{\frac{1}{2}\nu}\,\nu K_\nu(z). \qquad (1)$$

The numerical calculations were carried through for $\mu_0 = 5$, the approximate values of the A_m being taken as in 4.1 (4) (p. 454). The values of $2\Gamma\omega/\pi wv$, as given by 4.1 (9) with p equal to 4, are shown in the table below (Table III) and by the full line graph in fig. 2. The values of $F_{2,1}(\mu)$ or $\mu^2/(1+\mu^2) - T_{1,2}(2\mu)$, are also exhibited. It was not necessary to calculate $F_{2,5}(\mu)$, and the term in $F_{2,3}(\mu)$ never contributed more than 1 to the third figure.‡

* No tables are recorded in Watson's 'Bessel Functions' or in Mises, 'Verzeichnis berechneter Funktionentafeln,' Berlin, 1928.

† Watson, 'Bessel Functions,' § 10.71, p. 347.

‡ If the F are neglected, and we put $\mu^2/(1+\mu^2)$ for T in (1) with ν equal to $(4m+2)$ and z equal to $(4m+2)\,\mu$, we have an approximate formula to calculate K. This method, for example, gives $K_6(1\cdot 2)$ equal to $1197\cdot 4221$ instead of $1197\cdot 4227$, as given by Watson.

Table III.—Values of $F_{2,1}(\mu)$ and $2\Gamma\omega/\pi wv$ for a Four-bladed Propeller for $\mu_0 = 5$.

μ.	$F_{2,1}(\mu)$.	$2\Gamma\omega/\pi wv$.	μ.	$F_{2,1}(\mu)$.	$2\Gamma\omega/\pi wv$.
0·1	−0·0150	0·022	1·8	0·0157	0·750
0·2	−0·0341	0·066	2·0	0·0152	0·785
0·4	−0·0517	0·180	2·5	0·0115	0·848
0·6	−0·0432	0·300	3·0	0·0077	0·881
0·8	−0·0248	0·411	3·5	0·0050	0·887
1·0	−0·0075	0·506	4·0	0·0033	0·851
1·2	+0·0047	0·586	4·5	0·0021	0·714
1·4	+0·0117	0·652	4·8	0·0017	0·505
1·6	0·0149	0·706	5·0	0·0015	0

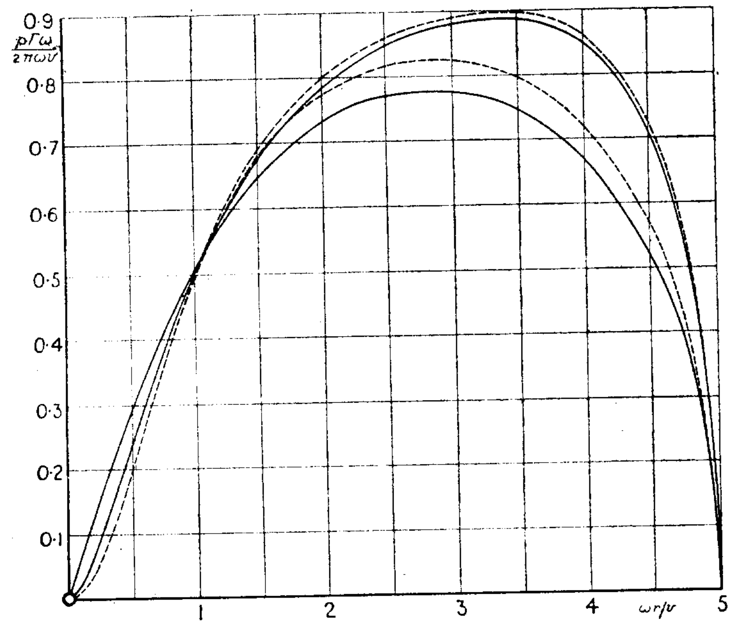

Fig. 2.—The Distribution of Circulation along a Propeller Blade for a Four-bladed and for a Two-bladed Propeller, when the Energy lost in the Slip Stream is a Minimum for a given Thrust.

The full line curves give the exact solution, the dotted curves the Prandtl approximation. In each case the curve with the greater maximum ordinate refers to a four-bladed propeller, the one with the less to a two-bladed propeller. The ordinates are $p\Gamma\omega/2\pi wv$, and the abscissæ $\omega r/v$, where p is the number of blades and the other symbols have the same meaning as before (fig. 1). The curves are drawn for $\omega R/v$ equal to 5. As before, each full line curve and the corresponding dotted curve are drawn for the same value of w.

The Prandtl approximation for a four-bladed propeller is

$$\frac{2\Gamma\omega}{\pi w v} = \frac{2}{\pi}\frac{\mu^2}{1+\mu^2}\cos^{-1} e^{-2f}, \tag{2}$$

where

$$f = (1+\mu_0^2)^{\frac{1}{2}}(1-\mu/\mu_0). \tag{3}$$

This is given by the broken line curve of fig. 2.

5. *The Application of the Solution.*

Let

$$K = \frac{p}{2\pi}\frac{\Gamma\omega}{wv}, \tag{1}$$

where p is the number of blades. We shall assume the distribution found in the previous sections, so that K is a given function of r/R for any given value of μ_0, or $\omega R/v$.

Let T be the thrust and Q the torque of the propeller, ρ the density of the fluid in which it is operating, and c_T and c_Q the non-dimensional coefficients defined by

$$c_T = T/\pi\rho R^2 v^2, \tag{2}$$

and

$$c_Q = \omega Q/\pi\rho R^2 v^3. \tag{3}$$

If η is the efficiency,

$$\eta = vT/\omega Q = c_T/c_Q. \tag{4}$$

Let u_θ and u_z be the circumferential and axial components of velocity far behind the propeller, as given by 3.5 (7), so that $u_\theta = -w\mu/(1+\mu^2)$, and $u_z = w\mu^2/(1+\mu^2)$. The velocities at the blade are half as much, if the contraction be neglected.

Let

$$r/R = x \quad \text{and} \quad w/v = \lambda. \tag{5, 6}$$

Then

$$\mu = \omega r/v = \mu_0 x. \tag{7}$$

Let us first neglect the profile drag of the blade sections. Then

$$\frac{dT}{dr} = p\rho\Gamma(rw + \tfrac{1}{2}u_\theta) = p\rho\Gamma\left(rw - \tfrac{1}{2}w\frac{\mu}{1+\mu^2}\right), \tag{8}$$

from which we find

$$\frac{dc_T}{dx} = 2\lambda Kx - \frac{\lambda^2 Kx}{1+\mu_0^2 x^2}, \tag{9}$$

so that

$$c_T = 2\lambda I_1 - \lambda^2 I_2, \tag{10}$$

where

$$I_1 = \int_0^1 Kx\,dx, \quad \text{and} \quad I_2 = \int_0^1 \frac{Kx\,dx}{1+\mu_0^2 x^2}. \tag{11, 12}$$

I_1 and I_2 are functions of μ_0, most easily found by plotting the integrand and using a planimeter. They can be plotted against μ_0 once and for all.

Further

$$\frac{dQ}{dr} = p\rho\Gamma r\,(v + \tfrac{1}{2}u_z) = p\rho\Gamma r\left(v + \tfrac{1}{2}w\,\frac{\mu^2}{1+\mu^2}\right), \tag{13}$$

from which we find

$$\frac{dc_Q}{dx} = 2\lambda K x + \frac{\lambda^2\mu_0^2 K x^3}{1+\mu_0^2x^2}, \tag{14}$$

so that

$$c_Q = 2\lambda I_1 + \lambda^2\mu_0^2 I_3, \tag{15}$$

where

$$\mu_0^2 I_3 = \mu_0^2\int_0^1 \frac{Kx^3\,dx}{1+\mu_0^2x^2} = I_1 - I_2. \tag{16}$$

Hence

$$\eta = \frac{I_1 - \frac{1}{2}\lambda I_2}{I_1 + \frac{1}{2}\mu_0^2\lambda I_3}. \tag{17}$$

Thus, working at any given value of μ_0, we find λ in terms of c_T from (10), and then η from (17). We thus get a series of curves of η against c_T for different values of μ_0, or of η against μ_0 for different values of c_T. The thrust and torque grading curves are given by (9) and (14).

The theory holds only for small values of λ, for which we have approximately

$$\eta = \frac{1}{1 + \frac{1}{2}\mu_0^2\lambda\, I_3/I_1 + \frac{1}{2}\lambda\, I_2/I_1} = \frac{1}{1+\frac{1}{2}\lambda},^{*} \tag{18}$$

and

$$\lambda = c_Q/2I_1. \tag{19}$$

We easily find that the energy lost per unit time is $\pi\rho v^3R^2\lambda^2I_1$.

If θ is the blade angle, the incidence of any section is $\theta - \phi$, where†

$$\tan\phi = \frac{v + \frac{1}{2}u_z}{r\omega + \frac{1}{2}u_\theta} = \frac{1}{\mu_0 x}\,\frac{1 + \frac{1}{2}\lambda\,\mu_0^2x^2/(1+\mu_0^2x^2)}{1 - \frac{1}{2}\lambda/(1+\mu_0^2x^2)} = \frac{1}{\mu_0 x}(1 + \tfrac{1}{2}\lambda)\text{ approx.} \tag{20}$$

We pass on to consider the effect of profile drag. Let ε be the ratio of drag to lift for a wing of infinite aspect ratio with the same section as the blade element at a distance r from the axis.† The contribution of the drag to the

* This form is preferable to $1 - \frac{1}{2}\lambda_1$, since $\mu_0^2 I_3$ is greater than I_2.

† The symbol ϕ is differently used in sections 2, 3 and 4. The symbol ϵ is differently used in sections 2 and 3.

element of thrust is then $-\varepsilon\, dQ_0/r$, where dQ_0 is the element of torque in the absence of profile drag. The contribution to the element of torque is similarly $\varepsilon r\, dT_0$, where dT_0 is the element of thrust in the absence of profile drag. Hence

$$\frac{dc_T}{dx} = \frac{dc_{T_0}}{dx} - \frac{\varepsilon}{\mu_0 x}\frac{dc_{Q_0}}{dx}, \tag{21}$$

and

$$\frac{dc_Q}{dx} = \frac{dc_{Q_0}}{dx} + \varepsilon\mu_0 x\frac{dc_{T_0}}{dx}. \tag{22}$$

dc_{T_0}/dx and dc_{Q_0}/dx are given by (9) and (14). Hence

$$\frac{dc_T}{dx} = 2\lambda Kx - \lambda^2\frac{Kx}{1+\mu_0^2x^2} - \frac{2\lambda}{\mu_0}\varepsilon K - \lambda^2\mu_0\frac{\varepsilon Kx^2}{1+\mu_0^2x^2}, \tag{23}$$

and

$$\frac{dc_Q}{dx} = 2\lambda Kx + \lambda^2\mu_0^2\frac{Kx^3}{1+\mu_0^2x^2} + 2\lambda\mu_0\varepsilon Kx^2 - \lambda^2\mu_0\frac{\varepsilon Kx^2}{1+\mu_0^2x^2}. \tag{24}$$

Consequently

$$c_T = 2\lambda I_1 - \lambda^2 I_2 - \frac{2\lambda}{\mu_0}I_4 - \lambda^2\mu_0 I_5, \tag{25}$$

and

$$c_Q = 2\lambda I_1 + \lambda^2\mu_0^2 I_3 + 2\lambda\mu_0 I_6 - \lambda^2\mu_0 I_5, \tag{26}$$

where

$$I_4 = \int_0^1 \varepsilon K\, dx, \tag{27}$$

$$I_5 = \int_0^1 \frac{\varepsilon Kx^2\, dx}{1+\mu_0^2x^2}, \tag{28}$$

and

$$I_6 = \int_0^1 \varepsilon Kx^2\, dx. \tag{29}$$

Then

$$\eta = \frac{c_T}{c_Q} = \frac{I_1 - \frac{1}{2}\lambda I_2 - I_4/\mu_0 - \frac{1}{2}\lambda\mu_0 I_5}{I_1 + \frac{1}{2}\lambda\mu_0^2 I_3 + \mu_0 I_6 - \frac{1}{2}\lambda\mu_0 I_5}. \tag{30}$$

For small values of λ and ε,

$$\eta = \frac{1}{1+\frac{1}{2}\lambda + I_7/I_1},$$

where

$$I_7 = I_4/\mu_0 + \mu_0 I_6 = \frac{1}{\mu_0}\int_0^1 \varepsilon K(1+\mu_0^2x^2)\, dx. \tag{32}$$

Thus the influence of friction is of order $\varepsilon\mu_0$, and friction has more influence on propellers the greater the ratio of tip speed to forward velocity. The determination of ε depends upon the incidence, and therefore on the angle ϕ. For some purposes it is advantageous to take a rough average of ε, and write

$$I_7 = \frac{\bar{\varepsilon}}{\mu_0} \int_0^1 K(1 + \mu_0{}^2 x^2)\, dx. \tag{33}$$

The integral is then a pure function of μ_0. Any losses dependent on the variation of the distribution of Γ from that taken here can be thrown into the ε term.

For accurate work we take a set of values of λ, calculate ϕ from (20), find ε and determine I_4, I_5 and I_6 graphically. The corresponding values of c_T are given by (25), and of η by (30). Hence we get the curve of η against c_T for the given blade and the value of μ_0 considered. We can get a set of curves for different values of μ_0, which can be turned into a set of curves of η against μ_0 for different values of c_T.

The above considerations are valid only if λ, or $c_T/2I_1$, is small. For moderate loads we get a better approximation if, following Prandtl, we replace v by $v + \frac{1}{2}w$. Again, according to Betz, 'Handbuch der Physik,' vol. 7, pp. 256–259 (1927), the contraction in the slip stream can be allowed for by considering a screw surface for which μ_0 is $\omega R/(v + w)$, so that the solution of the potential problem given here will still retain its usefulness. But there are still difficulties to be overcome in applying it, and a closer discussion must remain over.

My thanks are due to Prof. Betz, who suggested the problem to me.

Summary.

The distribution of circulation along a propeller blade when, for a given thrust, the energy lost in the slip-stream is a minimum, is calculated exactly and compared with the approximate Prandtl formula. Numerical values are given, for a two-bladed propeller, for various values of $\omega R/v$, where ω is the angular velocity, R the radius, and v the velocity of advance of the propeller. From these values the curves in fig. 1 were drawn. An example ($\omega R/v = 5$) was also computed numerically for a four-bladed propeller, and the result is shown in fig. 2. The distribution for any number of blades and any value of $\omega R/v$ can be worked out numerically by the method used, the work being easier the greater the number of blades and the greater $\omega R/v$. Formulæ for the fluid velocities far behind the propeller were found, from which numerical values can be worked out by the help of methods and numerical tables given.

Formulæ are set out for the distribution of thrust and torque, and for the efficiency, when the distribution of circulation is that calculated.

Appendix 1.

The Solution in Trigonometrical Series for a Rotating Lamina.

If the screw surface moves along its axis with velocity w and rotates with angular velocity $\omega w/v$ about its axis, its displacement is merely along its own surface. The fluid motion is therefore the same whether the surface moves along its axis with velocity w, or rotates about its axis with angular velocity ω_1, equal to $-\omega w/v$. If ω/v becomes zero while ω_1 remains finite, we fall back on the case of the rotating lamina, whose solution* is

$$\phi + i\psi = -\tfrac{1}{4}i\mathrm{R}^2\omega_1 e^{-2t}, \tag{1}$$

where $t = \xi + i\eta$, and ξ, η are elliptical co-ordinates given by

$$\tau = re^{i\zeta} = \mathrm{R}\cosh t. \tag{2}$$

The solution is easily put into the form of trigonometrical series. For

$$\mathrm{R}e^{-t} = \tau - (\tau^2 - \mathrm{R}^2)^{\frac{1}{2}}. \tag{3}$$

Hence

$$\begin{aligned}\mathrm{R}^2e^{-2t} &= 2\tau^2 - \mathrm{R}^2 - 2\tau(\tau^2 - \mathrm{R}^2)^{\frac{1}{2}} \\ &= 2\tau^2 - \mathrm{R}^2 \mp 2\tau\mathrm{R}i\left\{1 - \tfrac{1}{2}\frac{\tau^2}{\mathrm{R}^2} - \frac{1\,.\,1}{2\,.\,4}\frac{\tau^4}{\mathrm{R}^4} - \frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\frac{\tau^6}{\mathrm{R}^6}\right. \\ &\qquad\left. - \frac{1\,.\,1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8}\frac{\tau^8}{\mathrm{R}^8} - \dots\right\} \text{ for } |\tau| < \mathrm{R} \\ &= 2\tau^2 - \mathrm{R}^2 - 2\tau^2\left\{1 - \tfrac{1}{2}\frac{\mathrm{R}^2}{\tau^2} - \frac{1\,.\,1}{2\,.\,4}\frac{\mathrm{R}^4}{\tau^4} - \frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\frac{\mathrm{R}^6}{\tau^6} - \dots\right\} \\ &\qquad \text{for } |\tau| > \mathrm{R}. \end{aligned} \tag{4}$$

Hence

$$\begin{aligned}\phi &= \tfrac{1}{2}\omega_1 r^2 \sin 2\zeta \mp \tfrac{1}{2}\mathrm{R}^2\omega_1\left\{\frac{r}{\mathrm{R}}\cos\zeta - \tfrac{1}{2}\left(\frac{r}{\mathrm{R}}\right)^3\cos 3\zeta - \frac{1\,.\,1}{2\,.\,4}\left(\frac{r}{\mathrm{R}}\right)^5\cos 5\zeta\right. \\ &\qquad\left. - \frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\left(\frac{r}{\mathrm{R}}\right)^7\cos 7\zeta - \dots\right\} \text{ for } r \leqslant \mathrm{R} \\ &= -\tfrac{1}{2}\mathrm{R}\omega_1\left\{\frac{1\,.\,1}{2\,.\,4}\left(\frac{\mathrm{R}}{r}\right)^2\sin 2\zeta + \frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\left(\frac{\mathrm{R}}{r}\right)^4\sin 4\zeta + \dots\right\} \text{ for } r \geqslant \mathrm{R}.\end{aligned} \tag{5}$$

The discontinuity in ϕ is $\omega_1 r(\mathrm{R}^2 - r^2)^{\frac{1}{2}}$.

* Lamb, 'Hydrodynamics,' chap. iv, § 72.

Appendix 2.

The Solution of the Potential Problem for $\omega R/v$ small.

We return to the potential problem formulated in section 2, namely,

$$\left(\mu\frac{\partial}{\partial\mu}\right)^2\phi + (1+\mu^2)\frac{\partial^2\phi}{\partial\zeta^2} = 0, \tag{1}$$

$$\frac{\partial\phi}{\partial\zeta} = -\frac{\mu^2}{1+\mu^2} \text{ at } \zeta = 0 \text{ or } \pi \text{ for } 0 \leqslant r \leqslant \mathrm{R}, \tag{2}$$

grad $\phi = 0$ at $r = \infty$, and ϕ is a single-valued function.

We assume

$$\phi = \sum_{n=1}^{\infty} a_n \mathrm{I}_{2n}(2n\mu) \sin 2n\zeta + \sum_{m=0}^{\infty} b_m \frac{\mathrm{I}_{2m+1}(\overline{2m+1}\,\mu)}{\mathrm{I}_{2m+1}(\overline{2m+1}\,\mu_0)} \cos(2m+1)\zeta$$

$$\text{for } r \leqslant \mathrm{R},\ 0 \leqslant \zeta \leqslant \pi, \tag{3}$$

and

$$\phi = \sum_{n=1}^{\infty} c_n \frac{\mathrm{K}_{2n}(2n\mu)}{\mathrm{K}_{2n}(2n\mu_0)} \sin 2n\zeta \text{ for } r \geqslant \mathrm{R}. \tag{4}$$

Then

$$\left(\frac{\partial\phi}{\partial\zeta}\right)_{\zeta=0 \text{ or } \pi} = \sum_{n=1}^{\infty} 2na_n \mathrm{I}_{2n}(2n\mu). \tag{5}$$

Now

$$\frac{\mu^2}{1+\mu^2} = 2\sum_{n=1}^{\infty}(-1)^{n-1}\,\mathrm{I}_{2n}(2n\mu), \tag{6}$$

if μ is less than $0 \cdot 66 \ldots$ (Watson's 'Bessel Functions,' Chap. 17).

Hence, if $\mu_0 < 0 \cdot 66 \ldots$, we can take

$$a_n = (-1)^n/n. \tag{7}$$

The constants b_m and c_n are then to be found from the conditions of continuity at $\mu = \mu_0$, namely, the continuity of ϕ and of $\partial\phi/\partial r$. This can be done numerically as in 3.3, by expanding $\cos(2m+1)\zeta$ in a sine-series; or by assuming all but a finite number of the b_m and c_n to be zero, and equating the expressions for ϕ, and also for $\partial\phi/\partial r$, at a finite number of selected values of ζ between 0 and π. The convergence will be slow.

Appendix 3.

The Approximate Determination of the Constants a_m and c_n.

1. Let

$$\phi = \sum_{n=1}^{\infty} \frac{1}{2n}\left(\frac{r}{a}\right)^{2n} \sin 2n\theta - \sum_{m=0}^{\infty} \mathrm{A}_m \left(\frac{r}{a}\right)^{2m+1} \cos (2m+1)\,\theta$$

$$\text{for } r \leqslant a,\ 0 \leqslant \theta \leqslant \pi,$$

$$= \sum_{n=1}^{\infty} \mathrm{B}_r \left(\frac{a}{r}\right)^{2n} \sin 2n\theta \qquad \text{for } r \geqslant a. \tag{1}$$

For $r \leqslant a$, $-\pi \leqslant \theta \leqslant 0$, let ϕ be as in the first line with the sign of the second term changed.

Expand $\cos (2m+1)\theta$ in a sine-series for $0 \leqslant \theta \leqslant \pi$, and equate coefficients of $\sin 2n\theta$. We obtain

$$\mathrm{B}_n = \frac{1}{2n} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{2n}{4n^2 - (2m+1)^2} \mathrm{A}_m. \tag{2}$$

Differentiate the two expressions for ϕ with respect to r and repeat the process. This gives

$$-\mathrm{B}_n = \frac{1}{2n} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{2m+1}{4n^2 - (2m+1)^2} \mathrm{A}_m. \tag{3}$$

Eliminate B_n. Then

$$\sum_{m=0}^{\infty} \frac{\mathrm{A}_m}{2n - 2m - 1} - \frac{\pi}{4n} = 0. \tag{4}$$

The equations (2), (3) and (4) hold for all positive integral values of n.

The value of ϕ so found is the solution of the problem defined by

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{5}$$

$$\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)_{\theta = 0 \text{ or } \pi} = \frac{1}{r} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{2n} = \frac{r}{a^2 - r^2}, \tag{6}$$

grad $\phi = 0$ at $r = \infty$, ϕ is a single-valued function.

This problem can be solved in finite terms by means of elliptic co-ordinates ξ, η for which

$$t = \xi + i\eta, \tag{7}$$

$$z = re^{i\theta} = a \cosh t. \tag{8}$$

The boundary condition becomes

$$\frac{\partial \phi}{\partial \xi} = \cot \eta \quad \text{at} \quad \xi = 0, \tag{9}$$

and the solution is

$$\phi + i\psi = i \log (1 - e^{-2t}). \tag{10}$$

Now

$$e^{-2t} = 2z^2/a^2 - 1 - 2(z/a)(z^2/a^2 - 1)^{\frac{1}{2}}, \tag{11}$$

so that

$$\begin{aligned} \phi + i\psi &= i \log 2 + i \log [1 - z^2/a^2 + (z/a)(z^2/a^2 - 1)^{\frac{1}{2}}] \\ &= i \log 2 + \tfrac{1}{2} i \log (1 - z^2/a^2) - \sin^{-1} z/a \\ &= i \log 2 - \tfrac{1}{2} i (z^2/a^2 + z^4/2a^4 + z^6/3a^6 + \ldots) \\ &\quad - \left(\frac{z}{a} + \frac{1\,.\,1}{2\,.\,3}\frac{z^3}{a^3} + \frac{1\,.\,3}{2\,.\,4}\frac{1}{5}\frac{z^5}{a^5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{1}{7}\frac{z^7}{a^7} + \ldots\right). \end{aligned} \tag{12}$$

Thus

$$\begin{aligned} \phi &= \tfrac{1}{2}\left(\frac{r^2}{a^2}\sin 2\theta + \tfrac{1}{2}\frac{r^4}{a^4}\sin 4\theta + \ldots\right) \\ &\quad - \left(\frac{r}{a}\cos\theta + \frac{1\,.\,1}{2\,.\,3}\frac{r^3}{a^3}\cos 3\theta + \frac{1\,.\,3}{2\,.\,4}\frac{1}{5}\frac{r^5}{a^5}\cos 5\theta + \ldots\right), \end{aligned} \tag{13}$$

and the solution of the equation (4) is

$$A_0 = 1, \quad 3A_1 = \tfrac{1}{2}, \quad 5A_2 = \frac{1\,.\,3}{2\,.\,4}, \quad 7A_3 = \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}, \ \ldots \tag{14}$$

The solution of the equations 3.3 (6) is then

$$a_m = -\frac{\mu_0^{\,2}}{1 + \mu_0^{\,2}} A_m, \tag{15}$$

where the A_m are given by (14).

2. With the approximations described in 3.3, the equations 3.3 (3), p. 447, become

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(2m+1)\,a_m}{4n^2 - (2m+1)^2} = -c_n. \tag{16}$$

This is the condition that

$$\sum_{m=0}^{\infty}(2m+1)\,a_m \cos(2m+1)\,\theta = -\sum_{n=1}^{\infty} 2n\,c_n \sin 2n\theta \tag{17}$$

for $0 \leqslant \theta \leqslant \pi$.*

* This comes directly if we differentiate 3.1 (1) and 3.1 (17) term by term with respect to μ, equate the results for μ equal to μ_0, and make the approximations described in 3.3.

Omit temporarily the factor $-\mu_0^2/(1+\mu_0^2)$, and take

$$(2m+1)\,a_m = \frac{1\,.\,3\,.\,5\,\ldots\,(2m-1)}{2\,.\,4\,.\,6\,\ldots\,2m}. \tag{18}$$

Let

$$C = \sum_{m=0}^{\infty} (2m+1)\,a_m \cos(2m+1)\,\theta, \tag{19}$$

and

$$S = \sum_{m=0}^{\infty} (2m+1)\,a_m \sin(2m+1)\,\theta, \tag{20}$$

so that

$$C + iS = \frac{e^{i\theta}}{(1-e^{2i\theta})^{\frac{1}{2}}} = \pm i\,\frac{1}{(1-e^{-2i\theta})^{\frac{1}{2}}}$$

$$= \pm i\left(1 + \tfrac{1}{2}e^{-2i\theta} + \frac{1\,.\,3}{2\,.\,4}e^{-4i\theta} + \ldots\right). \tag{21}$$

Then

$$C = \pm\left(\tfrac{1}{2}\sin 2\theta + \frac{1\,.\,3}{2\,.\,4}\sin 4\theta + \ldots\right), \tag{22}$$

and

$$S = \pm\left(1 + \tfrac{1}{2}\cos 2\theta + \frac{1\,.\,3}{2\,.\,4}\cos 4\theta + \ldots\right). \tag{23}$$

Putting $\lambda = \frac{1}{2}\pi$ in S, we see that the upper sign must be taken. Restoring the factor $-\mu_0^2/(1+\mu_0^2)$, we have finally

$$c_n = \frac{\mu_0^2}{1+\mu_0^2}\,\frac{C_n}{2n} \tag{24}$$

where

$$C_1 = \tfrac{1}{2},\quad C_2 = \frac{1\,.\,3}{2\,.\,4},\quad C_3 = \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\ldots, \tag{25}$$